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Abstract

In the last years there has been a growing interest in the understanding of a vast variety of scale invariant and critical phenomena occurring in nature. Experiments and observations indeed suggest that many physical systems develop spontaneously power law behavior both in space and time. Pattern formation, aggregation phenomena, biological and geological systems, disordered materials, clustering of matter in the universe are just some of the fields in which scale invariance has been observed as a common basic feature. In this respect fractal geometry has changed the way we look at nature and it has expanded the frontiers of physical sciences to include a wide variety of strongly irregular systems and complex phenomena. The value and impact of fractals, however, is still rather controversial. In this lecture we discuss the real advancements as well as the present limitations of this field by presenting it along three distinct lines, which constitute evolutionary stages: (i) Fractal geometry as a *mathematical framework* that allows us to identify and characterize scale invariant properties in natural phenomena. (ii) The development of *physical models* for the spontaneous development of fractal structures in well defined physical phenomena. (iii) The attempts to construct *physical theories* that should provide a full understanding for the self-organized origin of fractal structures in various systems. The style of the present discussion will be colloquial but the references can give a clue for a more technical level.

1 Introduction

Statistical physics is undergoing a profound transformation. The introduction of new ideas, inspired by fractal geometry and scaling, irreversible and non-ergodic dynamical systems leading to self-organization and stochastic processes of various types, leads to a considerable enrichment of the traditional framework and provides efficient methods for characterising and understanding complex systems.

The physics of scale-invariant and complex systems is a novel field which is including topics from several disciplines ranging from condensed matter physics to geology, biology, astrophysics and economics [1]. This widespread interdisciplinarity corresponds to the fact that these ideas allow us to look at natural phenomena in a radically new and original way, eventually leading to unifying concepts independently of the detailed structure of systems.

In scale invariant phenomena, events and information spread over a wide range of length and time scales, so that no matter what is the size of the scale considered one always observes surprisingly rich structures. These systems, with very many degrees of freedom, are usually so complex that their large scale behaviour cannot be predicted from the microscopic dynamics. New types of collective behaviour arise and their understanding represents one of the most challenging areas in modern statistical physics.

The interest in this field has been largely due to two factors. First the emerging availability of high powered computers over the past decade has enabled to readily simulate complex and disordered systems. Second the cross disciplinary mathematical language for describing these phenomena evolving under conditions far from equilibrium has only become available in the past years. The study of critical phenomena in second order transitions introduced the concepts of scaling and power law behavior [2]. Fractal geometry [3] provided the mathematical framework for the extension of these concepts to a vast variety of natural phenomena.

The physics of complex systems, however, turned out to be effectively new with respect to critical phenomena. The theory of equilibrium statistical physics is strongly based on the ergodic hypothesis and scale invariance develops at the critical equilibrium between order and disorder. Reaching this equilibrium requires the fine tuning of various parameters. On the contrary, most of the scale-free phenomena observed in nature are *self-organized*, in the sense that they spontaneously develop from the generating dynamical process. One is then forced to seek the origin of the scale invariance in nature in the rich domain of nonequilibrium systems and this requires the development of new ideas and methods.

The realization that certain structures exhibit fractal properties does not tell us why this happens but it is crucial to formulate the right questions. The impact of fractals in physics can be assessed along three different lines of increasing complexity:

(a) Fractal geometry merely as a *mathematical framework* which leads to a reanalysis of known data that results in a revamping of long-standing points of view. This permits to include into the scientific areas many phenomena characterised by intrinsic irregularities which have been previously neglected because of the lack of an appropriate mathematical. The main examples of this type can be found in the geophysical and astrophysical data and in Section 3 we consider one example in detail. The possibility of extending these methods also to biological evolution in terms of complex adaptive systems is also an active field of research.

(b) The development of *physical models* for systems that exhibit fractal and Self-Organized Critical (SOC) behaviour. From a mathematical point of view the problems explored are paticularly difficult. Often they consist of iterative systems with many degrees of freedom and irreversible dynamics. Very little can be predicted a priori for systems of this complexity, even though sometimes they can be very easy to fomulate. In this respect computer simulations represent an essential method in the physics of complex and scale invariant systems. While the great majority of the theoretical activity is based upon "toy models" which barely resemble real nature, it is important to build a bridge between theory and real experiments and this another basic task of computer simulations. This implies the development of models with the properties of a greater realism and large scale simulations which can be used also in material characterization. A byproduct of this approach is the application of fractal concepts to the solution of particular experimental problems (oil industry, disordered materials, phase nucleation, crystal growth etc.)

(c) The construction of complete *physical theories* that allow us to understand the self-organized origin of fractal structures as well as all the other relevant properties in various physical systems and phenomena. At a phenomenological level, scaling theory, inspired to usual critical phenomena, has been successfully used. This is essential for the rationalization of the results of computer simulations and experiments. This method allows us to identify the relations between different properties and exponents and to focus on the essential ones. The situation is completely different in relation to the formulation of a microscopic fundamental theory. The theoretical approach is particularly difficult because the statistical physics of systems far from equilibrium lies far beyond the usual equilibrium theory. This implies that the time development is intrinsically irreversible and that it cannot be eliminated by some form of the ergodic hypothesis. In equilibrium statistical mechanics it is in fact possible to eliminate the specific dynamical evolution and to assign directly a Boltzmann weight to a given configuration. In the case of self-organized fractal structures this is usually not possible and a full knowledge of the dynamical history is necessary. This implies the development of theoretical concepts of novel type.

2 Scale invariance and intrinsic irregularity

Most of theoretical physics is based on analytical functions and differential equations. This implies that structures should be essentially smooth and irregularities are treated as single fluctuations or isolated singularities. The study of critical phenomena and the development of the Renormalization Group (RG) theory in the seventies was therefore a major breakthrough [1, 4]. One could observe and describe phenomena in which intrinsic self-similar irregularities develop at all scales and fluctuations cannot be described in terms of analytical functions. The theoretical methods to describe this situation could not be based on ordinary differential equations because self-similarity implies the absence of analyticity and the familiar mathematical physics becomes inapplicable. In some sense the RG corresponds to the search of a space in which the problem becomes again analytical. This is the space of scale transformations but not the real space in which fluctuations are extremely irregular. For a while this peculiar situation seemed to be restricted to the specific critical point corresponding to the competition between order and disorder. In the past years instead, the development of fractal geometry [3], has allowed us to realize that a large variety of structures in nature are intrinsically irregular and self-similar.

Mathematically this situation corresponds to the fact that these structures are singular in every point. This property can be now characterized in a quantitative mathematical way by the fractal dimension and other suitable concepts. However, given these subtle properties, it is clear that making a theory for the physical origin of these structures is going to be a rather challenging task. This is actually the objective of the present activity in the field [5].

The main difference between the popular fractals like coastlines, mountains, trees, clouds, lightning, etc. and the self-similarity of critical phenomena is that criticality at phase transitions occurs only with an extremely accurate fine-tuning of the critical parameters involved. In the more familiar structures observed in nature instead the fractal properties are self-organized, they develop spontaneously from the dynamical process. It is probably in view of this important difference that the two fields of critical phenomena and fractal geometry have proceeded

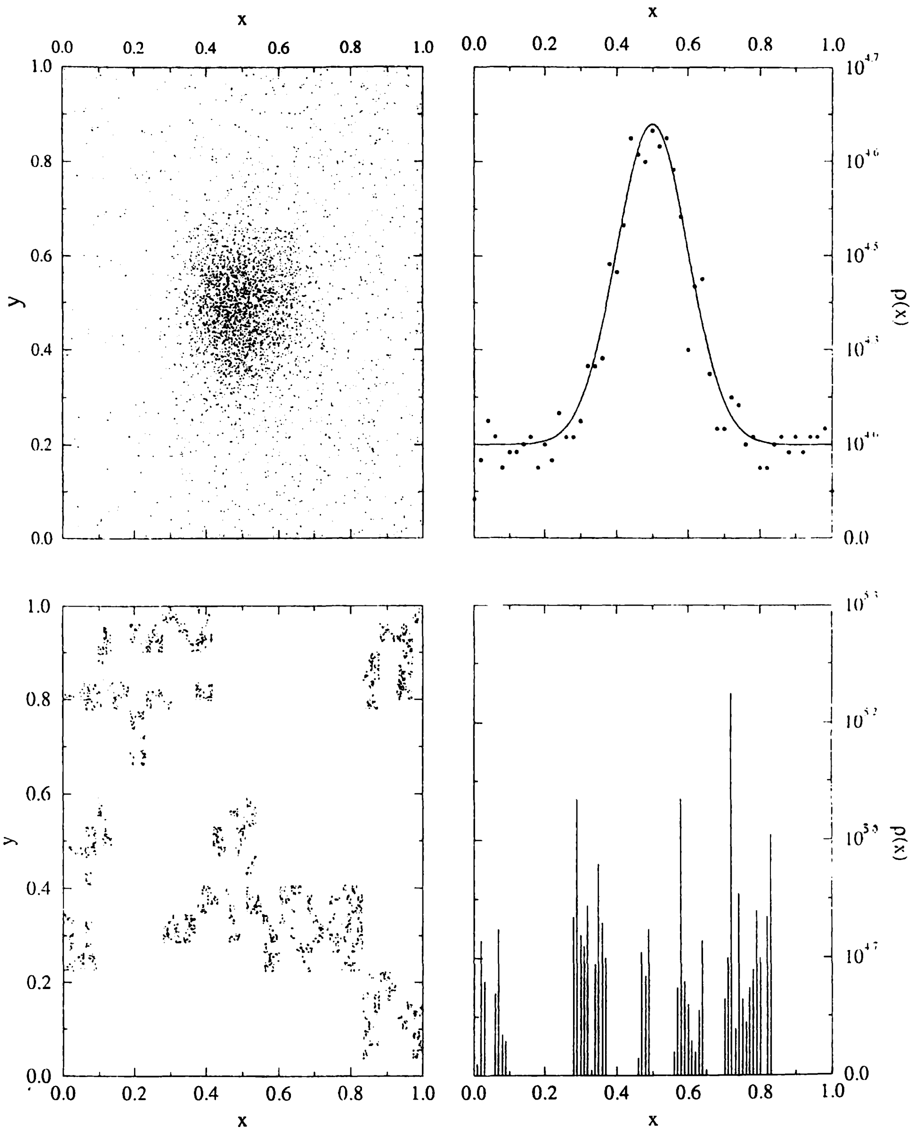

Figure 1: Example of analytical and nonanalytic structures. Top panels: (Left) A cluster in a homogenous distribution. (Right) Density profile. In this case the fluctuation corresponds to an enhancement the two-dimensional Euclidean space. (Right) Density profile. In this case the fluctuations are non-analytical and there is no reference value, i.e. the average density. The average density scales as a power law from any occupied point of the structure.

somewhat independently, at least at the beginning. The fact that we are traditionally accustomed to think in terms of analytical structures has crucial consequences on the type of questions we ask and on the methods we use to answer them. If one has never been exposed to the subtleness on nonanalytic structures, it is natural that analyticity is not even questioned. It is only after the above developments that we could realize that the property of analyticity can be tested experimentally and that it may or may not be present in a given physical system.

3 Fractal properties of the large-scale universe

In this section we discuss an example of the first category mentioned in the introduction in which the concept of Fractal Geometry, used as a mathematical tool, discloses new properties for the large-scale strucure of the universe and leads to fascinating and controversial perspectives.

The three-dimensional distribution of galaxies appears quite irregular and it consists of large structures and large voids. In the example shown in Figure 2 our galaxy is at the center and the empty slice corresponds to the galactic plane in which observations are difficult. Note that the picture is a projection (orthogonal) and this gives a smoothing effect to the eye. If one could rotate this picture as in a video the large structures and large voids would be better defined. Despite these structures the universe is believed to be homogeneous at large scale and this property is supposed to be in agreement with the data of Figure 2.

Some years ago we proposed a new approach for the analysis of galaxy and cluster correlations based on the concepts and methods of modern statistical physics. This led to the surprising result that galaxy correlations are fractal and not homogeneous up to the limits of the available catalogues. In the meantime many more red shifts have been measured and we have extended our methods also to the analysis of various other properties [6, 8].

The usual statistical methods, based on the assumption of homogeneity [9], appear therefore to be inconsistent for all the length scales probed until now. A new, more general, conceptual framework is necessary to identify the real physical properties of these structures, and theories should shift from "amplitudes" to "exponents" in the sense discussed in the previous section.

The new analysis shows that all the available data are consistent with each other and show fractal correlations (with dimension $D = 2$) up to the deepest scales probed until now (1000Mpc) [7, 8]. In these units, megaparsecs, the radius

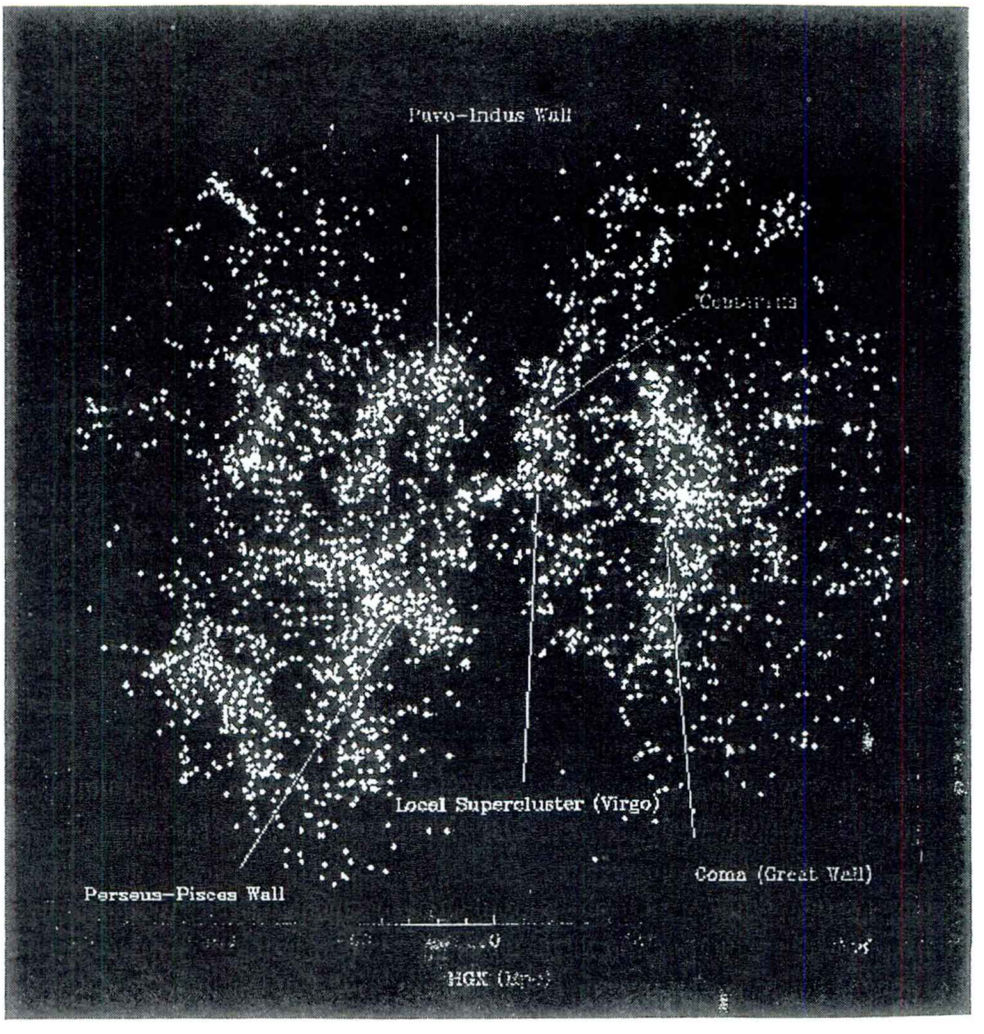

Figure 2: Three-dimensional distribution of galaxies around our galaxy (central point). The zone represented corresponds to about one tenth of the size of the entire universe.

of the entire universe is about 4000Mpc, while the size of a single galaxy (a point in our analysis) is about 0.01-0.1Mpc. The distribution of visible matter in the universe is therefore fractal and not homogeneous. In addition, the luminosity distribution is correlated with the space distribution in a specific way characterized by multifractal properties. These facts lead to fascinating conceptual implications about our knowledge of the universe and to a new scenario for the theoretical challenge.

This result has caused a large debate in the field [6] because it is in contrast with the usual assumption of large-scale homogeneity which is at the basis of most theories. Actually homogeneity represents much more than a working hypothesis for theory, it is often considered as a paradigm or principle and for some authors it is conceptually absurd even to question it [9].

For other authors instead, homogeneity is just the simplest working hypothesis and the idea that nature might actually be more complex is considered as extremely interesting [10]. These two points of view are not so different after all because, if something considered absurd becomes real, then it is indeed very exciting.

The problem is that these concepts touch directly the so-called Cosmological Principle (CP), which represents one of the landmarks of the field of cosmology. It is quite reasonable to assume that we are not in a very special point of the universe and to consider this as a principle, the CP. The usual mathematical implication of this principle is that the universe must be homogeneous [9]. This reasoning implies the hidden assumption of analyticity that often is not even mentioned. In fact the above reasonable requirement only leads to local isotropy. For an analytical structure this also implies homogeneity [10]. However, if the structure is not analytical, the above reasoning does not hold. For example a fractal structure has local isotropy but not homogeneity. In simple terms this means that all galaxies live in similar enviroments made of structures and voids (statistical isotropy). Therefore a fractal structure satisfies the CP in the sense that all the points are essentially equivalent (no center or special points) but this does not imply that these points are distributed uniformly.

The usual correlation analysis is performed by estimating at which distance (r_0) the density fluctuations are comparable to the average density in the sample. In practice this is done by considering the function $\xi(r) = < n(0)n(r) > / < n >^2 -1$, and by defining the characteristic length (r_0) as the point at which $\xi(r_0) = 1$. Now everybody agrees that there are fractal correlations at least at small scales. The important physical question is therefore to identify the distance λ_0 at which, possibly, the fractal distribution has a crossover into a homogeneous one. This would be the real correlation length beyond which the distribution can be approximated by an average density. The problem is therefore to understand the relation between r_0 and λ_0: are they the same or closely related or do they correspond to different properties?

This is actually a subtle point with respect to the concepts discussed in Section 2. In fact, if the galaxy distribution becomes really homogeneous at a scale λ_0 within the sample in question, then the value of r_0 is proportional to λ_0 and is related to the real correlation properties of the system.

If, on the other hand, the fractal correlations extend up to the sample limits, then the resulting value of r_0 has nothing to do with the real properties of the galaxy distribution but it is fixed just by the size of the sample [6].

Given this situation of ambiguity with respect to the real meaning of r_0, it is clear that the usual correlation study in terms of the function $\xi(r)$ is not the appropriate method to clarify these basic questions. The essential problem is that, by using the function $\xi(r)$, one defines the amplitude of the density fluctuations by normalizing them to the average density of the sample in question. This implies that the observed density should be the real one and it should not depend on the given sample or on its size apart from Poisson fluctuations. However, if the distribution shows long-range (fractal) correlations, this approach becomes meaningless. For example if one studies a fractal distribution with $\xi(r)$ a characteristic length r_0 will be identified, but this is clearly an artifact because the structure is characterized exactly by the absence of any defined length [6].

The appropriate analysis of pair correlations should therefore be performed using methods that can check homogeneity or fractal properties without assuming a priori either one. The simplest method to do this is to consider directly the conditional density $\Gamma(r) =< n(0)n(r) >$ without comparing it to the average density. This is not all however because one has also to be careful not to make hidden assumptions of homogeneity in the treatment of the boundary conditions [6, 8]. For these reasons the statistical validity of a sample is limited to the radius (Rs) of the largest sphere that can be contained in the sample.

The main data of our correlation analysis are collected in Figure 3 in which we report the conditional density as a function of distance for various galaxy catalogues. The properties derived from different catalogues are compatible with each other and show a power law decay (fractal correlations) for the conditional density from 1Mpc to 150Mpc without any tendency towards homogenization (flattening). Using also other data for which only a limited analysis is possible, one can see that the fractal behavior continues up to about 1000Mpc. (For a detailed discussion see [8]). This implies necessarily that the value of r_0 (derived from the $\xi(r)$ approach) is actually spurious and it will scale with the sample size Rs as discussed in detail in [8]. The behaviour observed corresponds to a fractal structure with dimension $D = 2$. A homogeneous distribution would correspond to a flattening of the conditional density which is never observed.

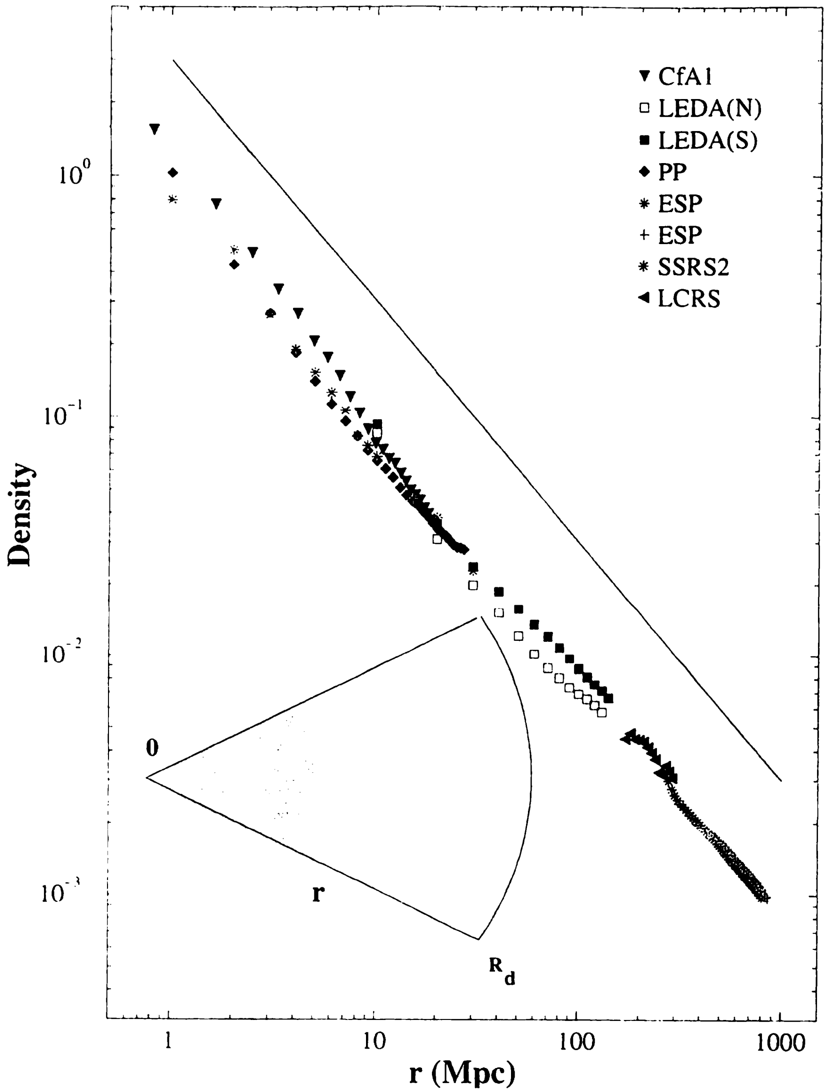

Figure 3: Correlation analysis for various three dimensional galaxy catalogues in the range 0.1 - 1000Mpc. The plot refers to the behavior of the conditional density as a function of distance. A reference line with slope -1 is also shown (i.e. fractal dimension $D = 2$). A constant density would correspond to a flat behavior.

It is important to remark that the usual correlation analyses have had a profound influence in the field in various ways [9]: first the various catalogues appear in conflict with each other. This has generated a strong mutual criticism about the validity of the data between different groups. In other cases the discrepancies observed have been considered as real physical problems for which various theoretical approaches have been proposed. These problems are, for example, the galaxy-cluster mismatch, luminosity segregation, the richness clustering relation and the linear and non-linear evolution of the perturbations corresponding to the "small" or "it large" amplitudes of fluctuations. We can now see that all this problematic is not real and it arises only from a statistical analysis based on inappropriate assumptions that do not find a correspondence in physical reality. It is also important to note that, even if the galaxy distribution would eventually become homogeneous at some large scale, the use of the above statistical concepts is anyhow inappropriate for the range of scales in which the system shows fractal correlations as those shown in Figure 3.

Up to now we have discussed galaxy correlations only in terms of the set of points corresponding to their position in space. Galaxies can be also characterized by their luminosity (related to their mass) and the luminosity distribution is then a full distribution and not a simple set. It is natural then to consider the possible scale invariant properties of this distribution. This requires a generalization of the fractal dimension and the use of the concept of multifractality [8]. A multifractal analysis shows that also the full distribution is scale invariant and this leads to a new and important relation between the Schechter luminosity distribution and the space correlation properties. This allows us to understand various morphological features (like the fact that large elliptic galaxies are typically located in large clusters) in terms of multifractal exponents. This leads also to a new interpretation of what has been called the luminosity segregation effect [8].

In summary our main points are:

(a) The highly irregular galaxy distributions with large structures and voids strongly point to a new statistical approach in which the existence of a well defined average density is not assumed a priori and the possibility of non-analytical properties should be addressed specifically.

(b) The new approach for the study of galaxy correlations in all the available catalogues shows that their properties are actually compatible with each other and they are statistically valid samples. The severe discrepancies between different

catalogues that have led various authors to consider these catalogues as not fair, were due to the inappropriate methods of analysis.
(c) The correct two-point correlation analysis shows well-defined fractal correlations up to the present observational limits, from 1 to 1000Mpc. with fractal dimension $D = 2$.
(d) The inclusion of the galaxy luminosity (mass) leads to a distribution which is shown to have well-defined multifractal properties. This leads to a new, important relation between the luminosity function and that galaxy correlations in space.

From the theoretical point of view, the fact that we have a situation characterized by self-similar structures implies that we should not use concepts which make reference to the average density or related properties. One cannot talk about "small" or "large" amplitudes for a self-similar structure because of the lack of a reference value like the average density. Physics should shift from "amplitudes" towards "exponents" and the methods of modern statistical Physics should be adopted. This leads to a new, fascinating situation, that has been uncovered by the introduction of the concepts of self-similarity and fractal geometry.

4 Fractal physical models

The key question is *how does nature produce fractal structures.* The first physical model that shed light on this question was the *Diffusion Limited Aggregation* (DLA) model of Witten and Sander [11] introduced in 1981. The model was inspired by the observation of growing aggregates that were found to exhibit fractal structures. One starts with a seed particle and introduces a new particle at some (large enough) distance R that executes a random walk on a lattice. When the particle reaches a site adjacent to the seed, it is frozen in that position and extends the seed. A new particle is then introduced until it touches the new seed and so on. The iteration of this simple algorithm produces structures of great complexity with a fractal dimension $D = 1.7$ (for planar growth). An interesting variant of DLA is the *Cluster-Cluster* aggregation model [12] where one starts with many particles executing random walks that are allowed to aggregate into clusters. Clusters of all sizes continue to execute random walks forming cluster aggregates and so on. Each cluster turns out to be fractal with a dimension that is lower than in the DLA model. In addition the distribution of cluster sizes exhibits power-law behavior. The Cluster-Cluster model captures the physics of dust or smoke clouds and colloids [13] as shown in Figure 4.

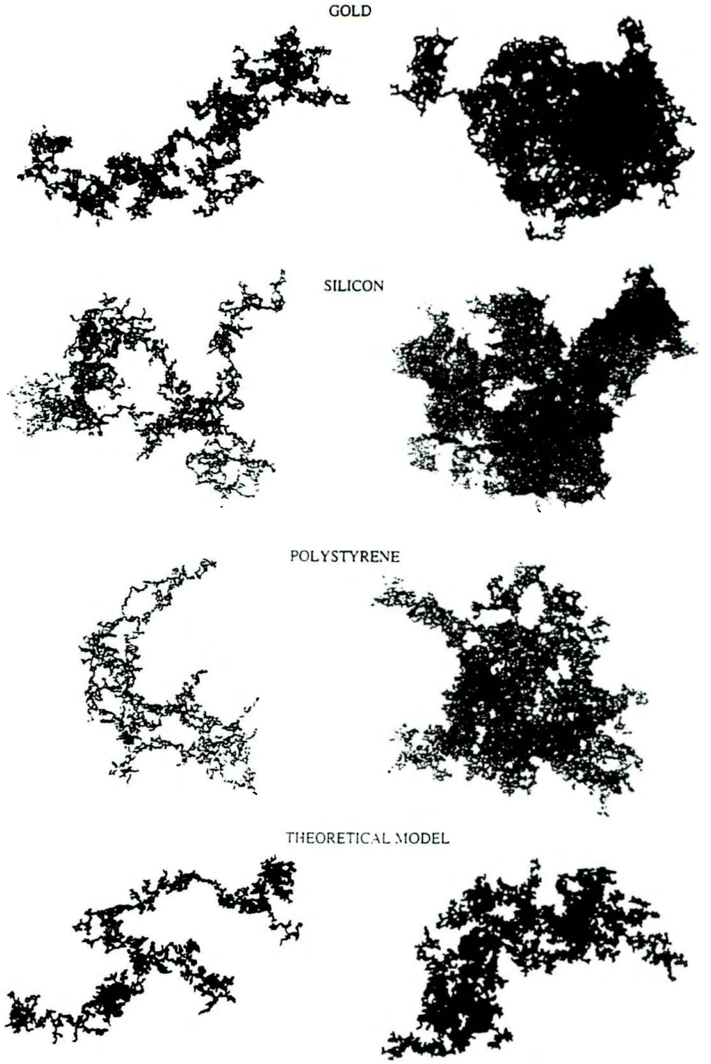

Figure 4: Real aggregation processes compared with the theoretical models. On the left we report the case of growth conditions in which the structure cannot rearrange after aggregation (low temperature). On the right, instead, structures may rearrange (high temperature). The first three cases correspond to real materials while the figures at the bottom correspond to the result of the theoretical model for the two growth regimes. (Courtesy of D. Weitz [12].)

In 1984 Niemeyer et al. introduced the Dielectric Breakdown Model (DBM) [14] inspired by discharges in gases (e.g. lightning). The discharge pattern is assumed to be composed of discrete points connected by bonds (see Figure 5) and

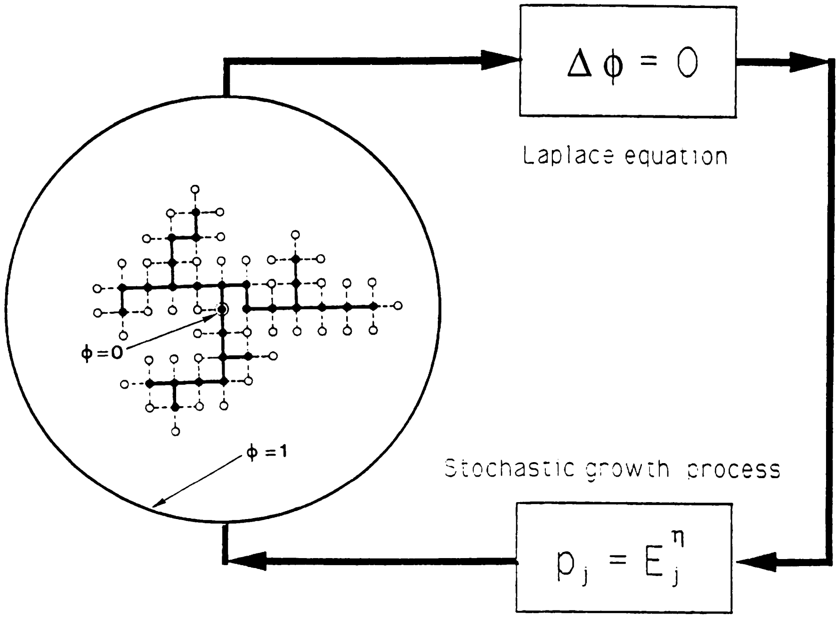

Figure 5: Schematic picture of DBM growth process. The grown structure is assumed to be equipotential. From the Laplace equation one can compute the local field for the bonds around the structure. The growth probability is related to the local field. A bond is selected and added to the structure and the process is then iterated.

the entire pattern at a given time is considered equipotential. At each perimeter point, a growth probability is assigned to be proportional to the local electric field E or to a power E^η. The electric field is determined from Laplace's equation for the electrical potential. The stochastic iteration of the model produces fractal structures with fractal dimensions that depend on h. In the case $h = 1$ one recovers the DLA structure. Apart from generalizing the DLA growth process the DBM illustrates the underlying mathematical properties in relation to partial differential equations like Laplace equation. This connection is quite surprising because usually Laplace equation produces smooth solutions: the potential at a given point is the average of the potential of the neighburing points. Here we see instead that a stochastic growth scheme in which the probabilities are defined by Laplace equation drives *spontaneously* the growth boundaries into a highly irregular fractal structure. In Figure 6 we show a fractal structure of the DLA/DBM in which the black and white stripes provide a visual impression of the variation of the electrical potential around the structure. A pair of black and white stripes corresponds to a change of a decade in the potential.

Figure 6: DLA/DBM cluster with potential stripes.

These "Laplacian" fractals are believed to capture the essential fractal properties of a variety of phenomena such as electrochemical deposition, dielectric breakdown, viscous fingering in fluids, the propagation of fractures and various properties of colloids [15].

The essential properties of these growth models are (for a detailed discussion see [5]):

— The growth process is *irreversible*. There is a growing interface and a "frozen" zone that will not be modified by further growth. The asymptotic properties and the fractal dimension refer to the frozen asymptotic structure.

— In order to assign a statistical weight to a structure it is necessary to know its entire *growth history*.

— The dynamics of these models evolves *spontaneously* into a fractal structure without the fine-tuning of any critical parameter, as is instead the case in ordinary critical phenomena.

— The degree of *universality* appears to be reduced with respect to equilibrium critical phenomena. For example in DBM the fractal dimension is a continuous function of the parameter h, but even for the standard DLA model, radial or cylindrical boundary conditions produce non-trivial differences.

The concept of spontaneous generation of complex or critical structures, also called *Self-Organized Criticality* (SOC), has been recently emphasized and investigated in the *sandpile models* introduced by Bak and coworkers [16]. To illustrate the basic ideas of SOC, they introduced a cellular automaton model of sandpiles. The random addition of sand grains drives the system towards a stationary state with a scale-free distribution of avalanches. As in the previous fractal growth models, also in this model criticality seems to emerge automatically without the fine-tuning of parameters. Because of the enormous conceptual power, SOC ideas have invaded rapidly throughout the sciences, from physics and geophysics to biology and economics, as a prototype mechanism to understand the manifestation of scale invariance and complexity in natural phenomena. It is interesting to compare in Table I the properties of these new models of fractal growth and SOC with those of standard critical phenomena represented by the Ising model.

Another model that was developed to simulate the displacement of a fluid in a porous medium is Invasion Percolation (IP) [15]. The porous medium is represented by a lattice where each bond has an assigned (quenched) value for its conductance. The dynamics of the fluid is to invade the bond with highest conductance within all its perimeter bonds. This model leads spontaneously to a fractal structure that is essentially identical to the percolating cluster of standard percolation [16]. The IP model, characterised by an extremal statistics, has recently inspired simple SOC models aimed at the description of the propagation of

<table>
<tr><th colspan="3">SELFSIMILARITY: PHYSICAL MODELS</th></tr>
<tr><th>Ising-Type (70's)</th><th>DLA/DBM (81)</th><th>Sandpile (87)</th></tr>
<tr><td>Equilibrium
Statistical
Mechanics

Ergodicity</td><td colspan="2">NON LINEAR, IRREVERSIBLE DYNAMICAL EVOLUTION.
Assigning the statistical weight of a structure requires the knowledge of its complete growth history.</td></tr>
<tr><td>Boltzmann Weight</td><td colspan="2">CRITICAL BEHAVIOR IS SELF-ORGANIZED
ATTRACTIVE FIXED POINT</td></tr>
<tr><td>Standard Critical behaviour
Fine Tuning: $T = T_c$</td><td></td><td></td></tr>
<tr><td>Repulsive Fixed Point
$\xi = (T - T_c)^{-\nu}$
Approach to the critical point</td><td>Asymptotically frozen fractal structure

Long range interactions (Laplacian)</td><td>Dynamical driven stationary state with avalanches of all sizes</td></tr>
<tr><td>$\Gamma(r) = \frac{1}{r^{(d-2+\eta)}}$</td><td colspan="2">Complex continuum limit: Lattice regularization seems to be essential</td></tr>
<tr><td>Anomalous dimension exactly at $T = T_c$</td><td>Problem: understand and compute the fractal dimension D</td><td>Problem: distribution of avalanche sizes $P(s) = s^{-\tau}$</td></tr>
<tr><td>Theory: Renormalization Group</td><td colspan="2">Theory: NEW CONCEPTS ARE NEEDED</td></tr>
</table>

Table 1: Comparison between the Ising model and two of the most popular models that generate fractal or scale invariant structures in a self-organized way.

irregular surfaces or interfaces in a disordered medium and of scale-free events in biological evolution. Extremal dynamics in a quenched medium is also the essential theoretical ingredient of the Bak and Sneppen model of biological evolution [17].

Another principal subject where fractals play an essential role is the study of interface growth in disordered systems e.g. Kardar Parisi Zhang (KPZ) equation [18]. If we consider the DBM model and eliminate the effect of the Laplace equation by setting $h = 0$, all the perimeter bonds have the same growth probability. This is the Eden model [14] that leads to compact structures with an irregular surface characterized by a critical exponent. These models of surface growth are meant to describe the deposition of particles, the propagation of chemical reaction or fire fronts, the interface between fluids or a fluid in a porous medium under appropriate conditions [19].

5 New theoretical concepts and self-organization

The physical models discussed in the previous sections illustrate a number of physical situations that can lead to the generation of fractal structures. Comparison with experimental data suggests that these models capture the essential physics of various phenomena that produce fractal structures in nature. Such models however do not constitute a physical theory, and this is the next step of our discussion.

From the theoretical point of view the idea of many authors is that DLA/DBM and the other SOC models pose questions of a new type for which it would be desirable to have a common theoretical scheme [20]. The attempts to use the theoretical concepts developed for critical phenomena like field theory and the RG have been quite problematic for these new phenomena. The basic differences with respect to equilibrium phase transition is that the dynamics is irreversible and self-organized. There is no ergodic principle and it is not possible to assign a Boltzmann weight to a configuration without knowing its entire growth history.

The theoretical effort in this field can be separated into phenomenological or scaling theories and microscopic theories. The first approach has been extensively developed in the past years and it consists in defining consistency relations between the assumed scaling properties of the system. This phenomenological approach is essential in the analysis of computer simulations to identify and extract the relevant essential information. The microscopic approach consists in a comprehensive understanding of all the SOC and fractal properties of the system directly from the knowledge of the microscopic dynamics. In some specific cases exact results can also be obtained. The development of a microscopic theory is an extremely difficoult task in which some interesting progress has been made but many fundamental questions are still open.

It was natural however to expect that some of the theoretical concepts developed for critical phenomena should also work for fractal models. A notable attempt in this direction was made by Kardar, Parisi and Zhang (KPZ) [18] who showed that the dynamics of the growing profile of the Eden model surface growth can be described by a stochastic differential equation for which field theory and RG methods can be succesfully applied. This approach corresponds to mapping the irreversible dynamics of the problem onto an equilibrium problem for the statistics of the profile. Various experiments of surface growth show however surface fluctuations with exponents that are higher than those predicted by the KPZ equation. This is probably due to quenched disorder that cannot be described in terms of an effective equilibrium problem [19].

This brings us to the crucial problem of fractal growth. We have seen that most fractal growth models like DLA, DBM, Cluter-Cluster aggregation, Invasion Percolation and the sandpile models are characterized by an intrinsically irreversible dynamics. As a result the statistical weight of a configuration can be defined only with the knowledge of its entire history. In other words the temporal evolution is just as important as the spatial correlations, which is not at all the case in equilibrium phase transitions. In the latter, the ergodic principle allows one to eliminate the temporal dynamics and assign a statistical weight for each configuration in terms of the Boltzmann factor. Another important difference is that most fractal structures are *self-organized*. For these and other more technical reasons like the absence of an upper critical dimension in some of these models the usual methods of field theory and RG did not lead very far for this class of models.

One attempt of constructing a physical theory for the self-organized fractals with irreversible dynamics is the Fixed Scale Transformation (FST) [5]. This approach combines a technique of lattice path integrals to take into account the irreversible dynamics with the study of the scale invariant dynamics inspired by the RG theory. This combination allows us to compute the pair correlations induced by the irreversible dynamics between block variables of arbitrary size. In this way it is possible to understand the origin of self-organization in fractal growth in terms of an attractive fixed point for the scale invariant dynamics and to compute analytically the fractal dimension. At the moment the FST framework seems to be the only general approach for the broad class of self-organized fractals and related phenomena. This method has been succesfully applied to DLA/DBM, to Cluster-Cluster aggregation, to fracture models, to Invasion Percolation and related models [5] and finally also to the sandpile models [21]. This situation supports therefore the conjecture [20] that DLA and the sandpile models pose questions of a new type for which it would be desirable to define a common theoretical scheme.

There are several other approaches that address similar issues for specific problems, e.g. the work of Nagatani and others [22] and of Halsey [23] for DLA, the elegant algebraic methods of Dhar et al [24], the field theory approaches of Kardar et al [25] and of Bak and coworkers [26] for certain properties of SOC models and the Run Time Statistics [27] to deal with problems with quenched disorder like IP and the Bak and Sneppen model.

6 Open problems and further developments

As we have mentioned there has been some relevant progress on the theoretical side with the introduction of new ideas and methods. However, many important questions remain open. The objective would be to develop these ideas into a general and systematic theoretical framework with microscopic predictive power in relation to fractal growth and SOC properties. It would also be important to clarity the relations between these new models and usual critical phenomena especially in relation to the properties of self-organization and the concept of universality. For example a crucial issue is the role of universality in fractal and SOC phenomena. In usual critical phenomena the same exponents that define the onset of magnetisation also describe the liquid vapour transition in water. This strong universality appears to be a characteristic of equilibrium systems. Self-organized systems, on the other hand, do not seem to exhibit the same degree of universality as the fractal dimension can be easily altered by relatively simple changes in the growth process. This reduced universality is sometimes viewed as a negative element because one is forced to describe specific systems instead of a single universal model. The truth is probably the opposite. Some theoretical concepts can be considered as general or universal, but the inherent diversity of the various models that have been studied adds another fascinating dimension in the intellectual search. After all, the SOC fractal structures we observe in nature are quite various and different from each other. The preliminary knowledge we have at the moment suggests that there are some universal principles but the specific properties depend on the specific process. It is possible that this has to do with the fact that the domain of irreversible phenomena is much broader than that of equilibrium statistics. The definition of the classes and laws for this broader area is certainly one of the main tasks of the theoretical effort in this field.

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